THE PREDICTIVE PERFORMANCE EVALUATION AND NUMERICAL EXAMPLE STUDY FOR THE PRINCIPAL COMPONENT TWO-PARAMETERS ESTIMATOR

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Abstract - In this paper, detailed comparisons are given between those estimators that can be derived from the principal component two-parameter estimator such as the ordinary least squares estimator, the principal components regression estimator, the ridge regression estimator, the Liu estimator, the \( r-k \) estimator and the \( r-d \) estimator by the prediction mean square error criterion. In addition, conditions for the superiority of the principal component two-parameter estimator over the others are obtained. Furthermore, a numerical example study is conducted to compare these estimators under the prediction mean squared error criterion.

Keywords - Prediction Mean Squared Error; Principal Component Two-parameter Estimator; \( r-k \) estimator; ridge regression estimator; Liu estimator; \( r-d \) estimator.

I. INTRODUCTION

Consider the following linear regression model (Yang et al., 2016),

\[ y = X\beta + \epsilon \]  (1)

where \( y \) is a vector of observed values on response variable with order \( n \), \( X \) is a \( n \times p \) matrix of \( n \) observations on \( p \) explanatory variables with full column rank, \( p \) denotes a vector of unknown parameters with order \( p \), \( \epsilon \) is an \( n \times 1 \) vector of disturbance assumed to be distributed with mean vector \( 0 \) and variance covariance matrix \( \sigma^2 I \), and \( I \) is an identity matrix of order \( n \times n \) (Yang and Huang, 2016).

Let \( T = (T, M_{p-r}) \) be the orthogonal matrix such that

\[ T'Xe = \Lambda \]

where \( 0 < r \leq p \), \( \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p) \), \( \Lambda_r = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_r) \), \( \Lambda_{p-r} = \text{diag}(\lambda_{r+1}, \lambda_{r+2}, \ldots, \lambda_p) \)

and \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \) denote the ordered eigenvalues of \( XX' \), then the principal component two-parameter estimator (PCTP) of \( \beta \) proposed by Chang and Yang (2012) can be written as

\[ \hat{\beta}_r(k,d) = T(T'XXT' + dI)^{-1}(T'XXT' + dI)T'Xy \]  (2)

where \( k > 0 \) and \( 0 < d < 1 \).

Let \( Z = XT \) and \( \alpha = T'\beta \), then model (1) can be rewritten as

\[ y = Z\alpha + \epsilon \]  (3)

For model (3), the PCTP estimator of \( \alpha \) can be written as

\[ \hat{\alpha}_r(k,d) = (Z'Z + I_r)^{-1}(Z'Z + dI)(Z'Z + kI)Z'y \]  (4)

Similarly, the ordinary least squares (OLS) estimator, the principal components regression (PCR) estimator proposed by Massy (1965), the ordinary ridge regression (ORR) estimator proposed by Kennard and Hoerl (1970), the Liu estimator proposed by Kejian (1993), the \( r-k \) estimator proposed by Baye and Parker (1984) and the \( r-d \) estimator proposed by Kaçiranlar and Sakallioglu (2001) of \( \alpha \), can be written as follows (Gama and Mason, 2017),

\[ \hat{\alpha}_\text{OLS} = (Z'Z)^{-1}Z'y = \Lambda^{-1}Z'y \]  (5)

\[ \hat{\alpha}_r = (Z'Z_r)^{-1}Z'_ry = \Lambda^{-1}Z'_ry \]  (6)

\[ \hat{\alpha}_k = (Z'Z + kI)^{-1}Z'_y = (\Lambda + kI)^{-1}Z'_y \]  (7)

\[ \hat{\alpha}_d = (Z'Z + I)^{-1}(Z'Z + dI)\hat{\alpha}_\text{OLS} \]  (8)

\[ \hat{\alpha}_r(k) = (Z'Z_r + kI)^{-1}Z'_y \]  (9)

\[ \hat{\alpha}_r(d) = (Z'Z_r + I_r)^{-1}(Z'Z_r + dI)\hat{\alpha}_r \]  (10)

where \( k > 0 \) and \( 0 < d < 1 \).

The OLS estimator is unbiased and has the smallest variance among all linear unbiased estimators. On the other hand, the PCR estimator, the ORR estimator, the Liu estimator, the \( r-k \) estimator, the \( r-d \) estimator and the PCTP estimator proposed to combat multicollinearity are biased. To compare these estimators, many researches have been done under the mean square error criterion or the mean square error matrix criterion. A slight analysis of the predictive performance has been done about these estimators. Friedman and Montgomery (1985), Özbey and...
Kaçiranlar (2015) and Dawoud and Kaçiranlar (2017), used a similar method proposed by Gunst and Mason (1979) to study the predictive performance of estimators, and they focused on the prediction of a new response, considering the predictive ability of an estimator in a particular observation, rather than focusing on the mean of \( y \). Following the same idea, we will study the predictive performance of the PCTP estimator compared to the OLS estimator, the PCR estimator, the ORR estimator, the Liu estimator, the \( r-k \) estimator and the \( r-d \) estimator under the prediction mean squared error (PMSE) criterion (Caron et al., 2017).

II. PREDICTION MEAN SQUARED ERROR CRITERION

The PMSE of a predictor \( \hat{\mu}_0 \) is defined as:

\[
PMSE = E(y_0 - \hat{\mu}_0)^2
\]  

(10)

where \( y_0 \) is the value to be predicted and \( \hat{\mu}_0 \) is the predicted value of \( y_0 \).

For the orthogonal model (3), letting \( z_0 \) represent the orthonormal point at which the prediction \( \hat{\mu}_0 \) is made and denoting \( J = PMSE = E(y_0 - \hat{\mu}_0)^2 \), by simple calculation we can get the PMSE of the OLS estimator, the PCR estimator, the ORR estimator, the Liu estimator, the \( r-k \) estimator, the \( r-d \) estimator and the PCTP estimator as follows.

\[
J_{OLS} = \sigma^2 \left( 1 + \sum_{i=1}^{r} \frac{z_i^2}{\lambda_i} \right) \tag{11}
\]

\[
J_r = \sigma^2 \left( 1 + \sum_{i=1}^{r} \frac{z_i^2}{\lambda_i^2} \right) \tag{12}
\]

\[
J_d = \sigma^2 \left( 1 + \sum_{i=1}^{r} \frac{z_i^2}{\lambda_i + k} \right) + k \left( \frac{\sum_{i=1}^{r} \frac{z_i^2}{\lambda_i + k}}{\lambda_i} \right) \tag{13}
\]

\[
J_{rk} = \sigma^2 \left( 1 + \sum_{i=1}^{r} \frac{z_i^2}{\lambda_i + k} \right) + (1 - d) \left( \frac{\sum_{i=1}^{r} \frac{z_i^2}{\lambda_i + k}}{\lambda_i + k} \right) \tag{14}
\]

\[
J_{rd} = \sigma^2 \left( 1 + \sum_{i=1}^{r} \frac{z_i^2}{\lambda_i + (1 + k)} \right) + (1 - d) \left( \frac{\sum_{i=1}^{r} \frac{z_i^2}{\lambda_i + (1 + k)}}{\lambda_i + (1 + k)} \right) \tag{15}
\]

\[
J_{PCTP} = \sigma^2 \left( 1 + \sum_{i=1}^{r} \frac{z_i^2}{\lambda_i + (1 + k)} \right) + (1 - d) \left( \frac{\sum_{i=1}^{r} \frac{z_i^2}{\lambda_i + (1 + k)}}{\lambda_i + (1 + k)} \right) \tag{16}
\]

III. COMPARISONS OF PMSE IN TWO-DIMENSIONAL SPACE

Considering a two-dimensional space, a single prediction point \([z_{01}, z_{02}]\) is predicted; the ratio \( z_{02}^2 / z_{01}^2 \) can be obtained and used for a reference point in their comparisons. The reason for considering two-dimensional space is to be able to make a theoretical comparison and to show it on a graph, where one estimator is better than another in one area, the predictive performance of the PCTP estimator will be discussed by the same approach. \( \alpha_i^2 \) is set to zero, because nonzero values of \( \alpha_i^2 \) increase only the intercept values for \( J_k, J_d, J_{r-k}, J_{r-d} \) and \( J_{PCTP} \), but leave the curves for \( J_{OLS} \) and \( J_r \) unchanged. The comparison of \( J_{PCTP} \) with \( J_k, J_d, J_{r-k}, J_{r-d} \) gives the following theorems.

Theorem 1

a. if \( \alpha_1^2 < \frac{\sigma_1^2}{\lambda_1^2} \), then \( J_{PCTP} < J_{OLS} \);

b. if \( \alpha_1^2 > \frac{\sigma_1^2}{\lambda_1^2} \), then \( J_{PCTP} < J_{OLS} \) if and only if

\[
\frac{z_{02}^2}{z_{01}^2} < f_1(\alpha_1^2), \quad \text{where}
\]

\[
f_1(\alpha_1^2) = \frac{\sigma_1^2 - \sigma_1^2(\lambda_1 + d)\lambda_1}{\alpha_1^2 - \frac{\sigma_1^2}{\lambda_1} - \frac{\sigma_1^2}{(\lambda_1 + d)\lambda_1}} \tag{18}
\]

Proof: Suppose PCTP estimator is better than OLS estimator in the sense of PMSE, then \( J_{PCTP} < J_{OLS} \), that is

\[
\sigma_1^2 \left( 1 + \frac{z_{02}^2}{z_{01}^2}(\lambda_1 + d)\lambda_1 \right) > \sigma_1^2 \left( 1 + \frac{z_{02}^2}{z_{01}^2} + \frac{z_{02}^2}{z_{01}^2} \right) \tag{19}
\]

Simplifying formula (19) we have

\[
\frac{z_{02}^2}{z_{01}^2} \left( \frac{\alpha_1^2 - \frac{\sigma_1^2}{\lambda_1} - \frac{\sigma_1^2}{(\lambda_1 + d)\lambda_1}}{\sigma_1^2} \right) < \frac{z_{02}^2}{z_{01}^2} \left( \frac{\sigma_1^2 - \frac{\sigma_1^2}{\lambda_1} - \frac{\sigma_1^2}{(\lambda_1 + d)\lambda_1}}{\sigma_1^2} \right) \tag{20}
\]

here \( z_{01}, z_{02} \) and \( \frac{\sigma_1^2}{\lambda_1} - \frac{\sigma_1^2}{(\lambda_1 + d)\lambda_1} \) are always positive, the magnitude of \( \alpha_1^2 \) depends on the value of \( \alpha_1^2 \), define

\[
\frac{\sigma_1^2 - \frac{\sigma_1^2}{\lambda_1} - \frac{\sigma_1^2}{(\lambda_1 + d)\lambda_1}}{\frac{\sigma_1^2}{\lambda_1}} \tag{21}
\]

as a function of \( \alpha_1^2 \), denoted as \( f_1(\alpha_1^2) \), there is a vertical asymptote at the point \( \alpha_1^2 = \frac{\sigma_1^2}{\lambda_1} \).

a) when \( \alpha_1^2 < \frac{\sigma_1^2}{\lambda_1} \), we will obtain \( \frac{z_{02}^2}{z_{01}^2} > f_1(\alpha_1^2) \),
since $\frac{z_{01}^2}{z_{01}^2}$ is greater than zero, and $f_1(\alpha_2^2)$ is always negative, the PCTP estimator is consistently better than the OLS estimator.

b) when $\alpha_2^2 > \frac{\sigma^2}{\lambda_1}$, we will obtain $\frac{z_{02}^2}{z_{01}^2} < f_1(\alpha_2^2)$,

and $f_1(\alpha_2^2)$ is greater than zero. If $\frac{z_{02}^2}{z_{01}^2} < f_1(\alpha_2^2)$ is true, the PCTP estimator is superior to the OLS estimator.

**Theorem 2:** For any value of $z_{01}^2, z_{02}^2, \alpha_2^2, k, d$ and $\lambda_1$, we have $J_{PCTP} < J_r$.

**Proof:** Suppose PCTP estimator is better than PCR estimator in the sense of PMSE, then $J_{PCTP} < J_r$, that is

$$\sigma^2 \left(1 + \frac{z_{02}^2}{z_{01}^2} \frac{(\lambda_1 + d^2)^2}{(\lambda_1 + 1)^2 (\lambda_1 + k^2)^2} \right) z_{01}^2 \alpha_2^2 < \sigma^2 \left(1 + \frac{z_{02}^2}{z_{01}^2} \frac{\lambda_1}{\lambda_1 + 1} \right)^2$$

Simplifying formula (22) we have

$$z_{02}^2 (\alpha_2^2 - \alpha_2^2) < \frac{z_{01}^2}{\lambda_1} \left( \frac{\sigma^2}{\lambda_1} - \frac{\sigma^2(\lambda_1 + d^2)^2}{(\lambda_1 + 1)^2 (\lambda_1 + k^2)^2} \right)^2$$

Because $z_{01}^2 > 0$ and $\frac{1}{\lambda_1} - \frac{(\lambda_1 + d^2)^2}{(\lambda_1 + 1)^2 (\lambda_1 + k^2)^2} > 0$ are always valid, for any values of $z_{01}^2, z_{02}^2, \alpha_2^2, k, d$ and $\lambda_1$, we have $J_{PCTP} < J_r$.

**Theorem 3**

a) if $\alpha_2^2 > \frac{\sigma^2}{\lambda_2 ((\lambda_2 + 1)^2 - (1-d)^2)}$, then $J_{PCTP} < J_d$;

b) if $\alpha_2^2 > \frac{\sigma^2(\lambda_2 + d^2)^2}{\lambda_2 ((\lambda_2 + 1)^2 - (1-d)^2)}$, then $J_{PCTP} < J_d$,

if and only if $\frac{z_{02}^2}{z_{01}^2} < f_2(\alpha_2^2)$, where:

$$f_2(\alpha_2^2) = \frac{\sigma^2(\lambda_2 + d^2)^2}{\lambda_2 ((\lambda_2 + 1)^2 - (1-d)^2)}.$$ (24)

**Proof:** Suppose PCTP estimator is better than Liu estimator in the sense of PMSE, then $J_{PCTP} < J_d$, that is

$$\frac{z_{01}^2}{z_{01}^2} \sigma^2 \lambda_2 (\lambda_2 + d^2)^2 + \frac{z_{02}^2}{z_{01}^2} \sigma^2(\lambda_2 + d^2)^2 + \frac{z_{02}^2}{z_{01}^2} \sigma^2(\lambda_2 + d^2)^2$$

$$\lambda_2 ((\lambda_2 + 1)^2 - (1-d)^2) \lambda_2 ((\lambda_2 + 1)^2 - (1-d)^2)$$

Simplifying formula (25) we have

$$\frac{z_{01}^2}{z_{01}^2} \sigma^2(\lambda_2 + d^2)^2 \left(\frac{\lambda_2}{\lambda_2 + 1} - \frac{\lambda_2}{\lambda_2 + 1} \right)$$

Fracturing formula (26) we define

$$f_2(\alpha_2^2) = \frac{\sigma^2(\lambda_2 + d^2)^2}{\lambda_2 ((\lambda_2 + 1)^2 - (1-d)^2)}$$

as a function of $\alpha_2^2$, denoted as $f_2(\alpha_2^2)$, here $z_{02}^2, z_{01}^2$ and $\frac{\sigma^2(\lambda_2 + d^2)^2}{\lambda_2 ((\lambda_2 + 1)^2 - (1-d)^2)}$ are always positive, the magnitude of $f_2(\alpha_2^2)$ depends on the value of $\alpha_2^2$, there is a vertical asymptote at the point $\frac{\sigma^2(\lambda_2 + d^2)^2}{\lambda_2 ((\lambda_2 + 1)^2 - (1-d)^2)}$.

Thus:

a) when $\alpha_2^2 > \frac{\sigma^2(\lambda_2 + d^2)^2}{\lambda_2 ((\lambda_2 + 1)^2 - (1-d)^2)}$, we can get $\frac{z_{02}^2}{z_{01}^2} < f_2(\alpha_2^2)$, because $\frac{z_{02}^2}{z_{01}^2}$ is always positive, $f_2(\alpha_2^2)$ are negative, the PCTP estimator is consistently better than the Liu estimator.

b) when $\alpha_2^2 > \frac{\sigma^2(\lambda_2 + d^2)^2}{\lambda_2 ((\lambda_2 + 1)^2 - (1-d)^2)}$, we can obtain $\frac{z_{01}^2}{z_{01}^2} < f_2(\alpha_2^2)$ and $f_2(\alpha_2^2)$ is greater than zero.

If $\frac{z_{02}^2}{z_{01}^2} < f_2(\alpha_2^2)$ is valid, then PCTP estimator is superior to the Liu estimator.

**Theorem 4**

a) if $\alpha_2^2 > \frac{\sigma^2}{\lambda_2 + 2k}$, then $J_{PCTP} < J_k$;

b) if $\alpha_2^2 > \frac{\sigma^2(\lambda_2 + d^2)^2}{\lambda_2 ((\lambda_2 + 1)^2 - (1-d)^2)}$, then $J_{PCTP} < J_k$, if and only if $\frac{z_{02}^2}{z_{01}^2} < f_3(\alpha_2^2)$, where:

$$f_3(\alpha_2^2) = \frac{\sigma^2}{\lambda_2} - \frac{\sigma^2(\lambda_2 + d^2)^2}{(\lambda_2 + k)^2 ((\lambda_2 + 1)^2 - (1-d)^2)}$$

(27)

**Proof:** Suppose PCTP estimator is better than ORR estimator in the sense of PMSE, then $J_{PCTP} < J_k$, that is

$$\frac{z_{01}^2}{z_{01}^2} \sigma^2 \lambda_2 (\lambda_2 + d^2)^2 + \frac{z_{02}^2}{z_{01}^2} \sigma^2(\lambda_2 + d^2)^2 + \frac{z_{02}^2}{z_{01}^2} \sigma^2(\lambda_2 + d^2)^2$$

$$\lambda_2 ((\lambda_2 + 1)^2 - (1-d)^2) \lambda_2 ((\lambda_2 + 1)^2 - (1-d)^2)$$

Simplifying formula (28) we have
\[
\frac{z_{\alpha}^2}{z_{01}^2} = \left(\frac{\sigma^2 \lambda - \sigma^2 \lambda (\lambda + d)^2}{(\lambda + k)^2 (\lambda + 1)^2}\right) / \left(\alpha^2 - \frac{\sigma^2}{\lambda + 2k}\right)
\]

(29)

define

\[
\left(\frac{\sigma^2 \lambda - \sigma^2 \lambda (\lambda + d)^2}{(\lambda + k)^2 (\lambda + 1)^2}\right) / \left(\alpha^2 - \frac{\sigma^2}{\lambda + 2k}\right)
\]

as a function of \(\alpha^2\), denote as \(f_\delta(\alpha^2)\), and there is a vertical asymptote at point \(\alpha^2 = \frac{\sigma^2}{\lambda + 2k}\). Because the value of \(\frac{\sigma^2 \lambda - \sigma^2 \lambda (\lambda + d)^2}{(\lambda + k)^2 (\lambda + 1)^2}\) is always greater than zero, the value of \(f_\delta(\alpha^2)\) determines the value of \(\alpha^2\). Thus

a) when \(\alpha^2 < \frac{\sigma^2}{\lambda + 2k}\), then \(\frac{z_{\alpha}^2}{z_{01}^2} > f_\delta(\alpha^2)\), because \(\frac{z_{\alpha}^2}{z_{01}^2}\) is always positive, for \(\alpha^2 < \frac{\sigma^2}{\lambda + 2k}\), the \(f_\delta(\alpha^2)\) are negative, PCTP estimator is consistently better than the OLS estimator.

b) when \(\alpha^2 > \frac{\sigma^2}{\lambda + 2k}\), then \(\frac{z_{\alpha}^2}{z_{01}^2} < f_\delta(\alpha^2)\) , if \(\frac{z_{\alpha}^2}{z_{01}^2} s_f(\alpha^2)\) is true, the PCTP estimator is better than the OLS estimator.

**Theorem 5:** For any value of \(z_{01}^2, z_{02}^2, \alpha^2, k, d\) and \(\lambda_1\), we have \(J_{PCTP} < J_{r-k}\).

**Proof:** Suppose PCTP estimator is better than \(r-k\) estimator in the sense of PMSE, then \(J_{PCTP} < J_{r-k}\). That is

\[
\sigma^2 z_{\alpha}^2(\lambda + d)^2 \lambda_1 (\lambda + 1)^2 < \sigma^2 z_{01}^2 \lambda_1 (\lambda + k)^2 + \sigma^2 \alpha^2 (\lambda + d)^2 \lambda_1 (\lambda + k)^2
\]

(30)

By calculating we found that

\[
\sigma^2 \frac{z_{\alpha}^2 \lambda_1}{(\lambda + k)^2 (\lambda + 1)^2} > \sigma^2 \frac{z_{01}^2 \lambda_1}{(\lambda + k)^2 (\lambda + 1)^2} \alpha^2
\]

is always true, so for any values of \(z_{01}^2, z_{02}^2, \alpha^2, k\) and \(d\), we have

\(J_{PCTP} < J_{r-k}\).

**Theorem 6:** For any value of \(z_{01}^2, z_{02}^2, \alpha^2, k, d\) and \(\lambda_1\), we have \(J_{PCTP} < J_{r-d}\).

**Proof:** Suppose PCTP estimator is better than \(r-d\) estimator in the sense of PMSE, then \(J_{PCTP} < J_{r-d}\). That is:

\[
\sigma^2 z_{\alpha}^2(\lambda + d)^2 \lambda_1 (\lambda + 1)^2 < \sigma^2 z_{01}^2 \lambda_1 (\lambda + k)^2 + \sigma^2 \alpha^2 (\lambda + d)^2 \lambda_1 (\lambda + k)^2
\]

(31)

Simplifying formula (31) we have

\[
\sigma^2 \frac{z_{\alpha}^2 (\lambda + d)^2 \lambda_1}{(\lambda + 1)^2 (\lambda + k)^2} > \sigma^2 \frac{z_{01}^2 \lambda_1}{(\lambda + 1)^2 (\lambda + k)^2} \alpha^2
\]

(32)

By calculating we found that \(\sigma^2 z_{\alpha}^2(\lambda + d)^2 \lambda_1 (\lambda + 1)^2 < \sigma^2 z_{01}^2 \lambda_1 (\lambda + k)^2 (\lambda + 1)^2 (\lambda + k)^2\) is always valid, so for any values of \(z_{01}^2, z_{02}^2, \alpha^2, k, d\) and \(\lambda_1\), we have \(J_{PCTP} < J_{r-d}\).

**IV. NUMERICAL EXAMPLES**

In this section, a numerical example is given to illustrate the theoretical results. Let us recall the example given by Friedman and Montgomery (1985), Özbey and Kaçiranlar (2015). They set ridge parameter \(k = 0.1\), Liu parameter \(d = 0.9\), \(\lambda_1 = 1.95\), \(\lambda_2 = 0.05\) and use \(\alpha_2 = 0.95\) to indicate the degree of collinearity. In this article, we use the same numerical example to verify our theoretical results and set \(\sigma^2 = 1\). Because the ratio \(z_{\alpha}^2 / z_{01}^2\) and \(\alpha^2\) are always positive, we only consider the situation in the first quadrant. The numerical simulation results are summarized as follows. From (18), we have

\[
f_\delta(\alpha^2) = \frac{0.07973608}{\alpha^2 - 20}
\]

(33)

**Figure 1.** Compare of the PMSE for PCTP and OLS.

According to (33) and Fig. 1, we can see there is a vertical asymptote at point 20. When \(\alpha^2\) less than 20, the PCTP estimator is consistently better than the OLS estimator. When \(\alpha^2\) greater than 20, there will be two situations. If the value of \(z_{\alpha}^2 / z_{01}^2\) is less than the value of \(f_\delta(\alpha^2)\), the PCTP estimator is better than the OLS estimator. On the contrary, the OLS estimator is superior to the PCTP estimator.
From (23), we have
\[
\frac{1}{1.95} \left( \frac{(1.95 + 0.9) \times 1.95}{(1.95 + 1)^2 (1.95+0.1)} \right) = 0.07973608 > 0
\]  \hspace{1cm} (34)

According to formula (34) and Fig. 2, we can get for any value of \( z_{01}^2, z_{02}^2, \alpha_2^2, k, d \) and \( \lambda_1 \), the PCTP estimator is consistently superior to the PCR estimator.

From (24), we have
\[
\frac{f_1(\alpha_2^2)}{\alpha_2^2 - 16.52174} = \frac{0.04555786}{\alpha_2^2 - 16.52174} \]  \hspace{1cm} (35)

Formula (35) and Fig. 3 show that the function \( f_1(\alpha_2^2) \) has a vertical asymptote at point 16.52174, we can see from picture 3 when \( \alpha_2^2 < 16.52174 \) is valid, the PCTP estimator is uniformly better than the Liu estimator. When \( \alpha_2^2 > 16.52174 \), there are two situations. If the value of \( \frac{z_{02}^2}{z_{01}^2} \) is greater than the value of \( f_1(\alpha_2^2) \), the OLS estimator is better than the PCTP estimator. On the contrary, the PCTP estimator is superior to the OLS estimator.

From (27), we have
\[
\frac{f_1(\alpha_2^2)}{\alpha_2^2 - 4} = \frac{0.03092508}{\alpha_2^2 - 4} \]  \hspace{1cm} (36)

From (30), we get
\[
\frac{1.95}{2.05} \left( \frac{(1.95 + 0.9) \times 1.95}{(1.95 + 1)^2 (1.95+0.1)} \right) = 0.3120501 > 0
\]  \hspace{1cm} (37)

According to formula (37) and Fig. 5, for any value of \( z_{01}^2, z_{02}^2, \alpha_2^2, k, d \) and \( \lambda_1 \), we have the PCTP estimator is consistently superior to the \( r-k \) class estimator.

From (32), we get
\[
\frac{(\lambda_1 + d)^2}{\lambda_1^2} - \frac{(\lambda_1 + d)^2\lambda_1}{(\lambda_1 + 1)^2(\lambda_1 + k)^2} = 0.3266828 > 0
\]  \hspace{1cm} (38)

From Fig. 6 we can see that the comparison result of the \( r-d \) estimator is consistent with the comparison result of the \( r-k \) estimator and PCR estimator, that is, for any value of \( z_{01}^2, z_{02}^2, \alpha_2^2, k, d \) and \( \lambda_1 \), the PCTP estimator is consistently superior to the \( r-d \) class estimator.
V. CONCLUSIONS

In this paper, we investigate the predictive performance of PCTP estimator compared to the OLS estimator, the PCR estimator, the Liu estimator, the ORR estimator, the \( r-k \) estimator and the \( r-d \) estimator. The conditions that the PCTP estimator is superior to each estimator in the PMSE sense are found. It can be seen from the numerical example that for any value of \( \alpha_1, \alpha_2, k, d \) and \( \lambda_k \), the PCTP estimator is superior to the PCR estimator, the \( r-k \) class estimator and the \( r-d \) estimator. All the numerical example results are consistent with the theoretical result.

ACKNOWLEDGEMENTS

This work was supported by Guizhou Science and Technology Department (Grant No: Qian Science [2017]1083), Guizhou Provincial Education Department (Grant No: Qian Science co-JG word LKM [2015]014) and High-level Innovative Talents Project of Guizhou Province.

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Received: December 15th 2017
Accepted: June 30th 2018
Recommended by Guest Editor
Juan Luis García Guirao