TENSORIAL EQUATIONS FOR THREE-DIMENSIONAL LAMINAR BOUNDARY LAYER FLOWS

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Abstract— **Tensorial equations are derived for a laminar and attached boundary layer flow with null pressure gradient in the normal direction to a smooth three-dimensional surface. An incompressible, isothermal and viscous fluid of Newtonian type is assumed. Covariant derivatives in the three dimensional Euclidean domain are employed, where the surface curvature terms are implicitly included in the Christoffel symbols with the aim of writing the boundary layer equations in an invariant form irrespective to the particular choice of the coordinate system. These equations are covariant under a linear coordinate transformation on the two surface coordinates, and a scaling along the normal direction to the surface. As a test case, the boundary layer near a sphere in an axisymmetrical steady flow is numerically computed using a pseudo-spectral approach.**

Keywords— **boundary-layer equations, laminar steady flow, incompressible viscous fluid, three-dimensional surfaces, tensor analysis.**

I. INTRODUCTION

As it is well known, mechanical devices may require the calculation of fully three-dimensional boundary layers (e.g. see Schlitchting and Gersten, 2004), as those associated with flow inside turbomachines (Lakshminarayana, 1995), horizontal-axis wind turbine blades (Prado, 1995), laminar flow technology (Stock, 2006) or aerospace technology (Dwoyer *et al.*, 1978), among other cases. The laminar boundary layer equations for three-dimensional surfaces are well-known (e.g. see Cebeci and Cousteix, 1999; Dey, 2001). In any case, there are curvature effects that do not disappear as they do in the two-dimensional case (Reed and Lin, 1993). In fact, these curvature effects are present through the (local) principal curvatures of the surface. If one of them is null the boundary layer equations are independent of the curvature effects, as in planes, yawed infinite cylinders or wings (Pai, 1956).

There is nothing special about a particular system of coordinates, so a physical or geometrical law behind the equations for the boundary layer flow should be independent of them. The related equation is often first presented in Cartesian coordinates, and its differential form is typically presented in a coordinate independent system using vector notation based on the nabla operator (German, 2007). With the coordinate invariant form known, the solution is usually approached by selecting a particular coordinate system, e.g. Cartesian, cylindrical, or spherical ones. In order to determine a differential equation in a selected coordinate system, the nabla operator is expressed in terms of partial derivatives with respect to the chosen coordinates, or alternatively, the related operators are obtained through a tedious coordinate transformation from the well known Cartesian form. In this way, coordinate independent expressions of these equations can be cast in a specific form from which analytical and numerical solutions can be pursued. An alternative although equivalent approach to posing coordinate invariant equations is to employ a tensorial representation. As it is well known, a tensor equation is a coordinate invariant equation where the monomials have to be tensors of a same order and, in component format, all terms must contain the same free indices. Thus, tensorial equations, like those written in terms of the nabla operator, are form invariant, or covariant, with respect to regular curvilinear coordinate transformations. As in the case with the nabla form, operations such as the gradient and divergence are well defined. However their representation is in terms of the resource of tensor calculus. Besides the fact that both approaches are equivalents, the tensor approach often provides a more tractable and compact method for dealing with transformations among regular coordinate systems, since the relationships between the differential equation and the geometry can be a bit more clean than the nabla operator approach. For instance, a differential equation can be cast in terms of a particular set of coordinates through the specification of the