PARTIAL DIFFERENTIAL EQUATIONS FOR MISSING BOUNDARY CONDITIONS IN THE LINEAR-QUADRATIC OPTIMAL CONTROL PROBLEM

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***Abstract*— New equations involving the unknown final states and initial costates corresponding to families of LQR problems are found, and their solutions are computed and validated. Having the initial values of the costates, the optimal control can then be constructed, for each particular problem, from the solution to the Hamiltonian equations, now achievable through on-line integration. The missing boundary conditions are obtained by solving (off-line) two uncoupled, first-order, quasi-linear, partial differential equations for two auxiliary $n \times n$ matrices, whose independent variables are the time-horizon duration T and the final-penalty matrix S. The solutions to these PDEs give information on the behavior of the whole two-parameter family of control problems, which can be used for design purposes. The mathematical treatment takes advantage of the symplectic structure of the Hamiltonian formalism, which allows to reformulate one of Bellman's conjectures related to the "invariant-imbedding" methodology. Results are tested against solutions of the differential Riccati equations associated with these problems, and the attributes of the two approaches are illustrated and discussed.**

***Keywords*— optimal control, linear-quadratic problem, first order PDEs, boundary-value problems, Riccati equations.**

I. INTRODUCTION

The linear-quadratic regulator (LQR) problem is probably the most studied and used in the state-space optimal control literature. The main line of work in this direction has evolved around the algebraic (ARE, for infinite-horizon problems) and differential (DRE, for finite-horizon ones) Riccati equations. When expressed in $2n$-phase space, i.e. introducing the costate (the spacial derivative of the value function), the dynamics of the optimal control problem takes the form of the classical Hamilton's equations of fundamental Physics.

Since early sixties, Hamiltonian formalism has been at the core of the development of modern optimal control theory (Pontryagin *et al.,* 1962). When the problem concerning an n-dimensional system and an additive cost objective is regular (Kalman *et al.*, 1969), i.e. when the Hamiltonian of the problem can be uniquely optimized by a control value u^0 depending on the remaining variables (t,x,λ), then a set of $2n$ ordinary differential equations (ODEs) with a two-point boundary-value condition, known as Hamilton's (or Hamiltonian) equations (HE), has to be solved. This is often a rather difficult numerical problem. For the linear-quadratic regulator (LQR) with a finite horizon there exist well known methods (see for instance Sontag, 1998) to transform the boundary-value problem into an initial-value one. In the infinite-horizon, bilinear-quadratic regulator and change of set-point servo, there is a recent attempt to find the missing initial condition for the costate variable, which allows to integrate the equations on-line with the underlying control process (Costanza and Neuman, 2006).

Hamiltonian systems (modelled by a $2n$-dimensional ODE whose vector field can be expressed in terms of the partial derivatives of an underlying "total energy" function -called "the Hamiltonian"-, constant along trajectories), are key objects in Mathematical Physics. The ODEs for the state and costate of an optimal control problem referred above constitute a Hamiltonian system from this general point of view. Richard Bellman has contributed in both fields, but was particularly interested in symplectic systems coming from Physics (see for instance Abraham and Marsden, 1978) when he devised a partial differential equation (PDE) for the final value of the state $x(t_f)=r(T,c)$ as a function of the duration of the process $T=t_f-t_0$, and of the final value imposed to the costate $\lambda(t_f)=c$ (one of the boundary conditions, the other being the fixed initial value of the state $x(t_0)=x_0$, see Bellman and Kalaba, 1963). Bellman exploited in that case ideas common to the "invariant imbedding" numerical techniques, also associated with his name.

In Costanza (2008) the invariant imbedding approach is generalized and proved for the one-dimensional nonlinear-quadratic optimal control situation, where the final value of the costate depends on the final value of the state, i.e. $c=c(r)$. The procedure followed in this proof induces another PDE for the initial value σ of the costate $\lambda(t_0)$, which was actually the main concern from the optimal control point of view. The first-order quasilinear equation for σ developed here is new. It can be integrated after the PDE for the final