A METHOD FOR SOLVING AN INVERSE PROBLEM WITH UNKNOWN PARAMETERS FROM TWO SETS OF RELATIVE MEASUREMENTS

J.R. VEGA,† G.L. FRONTINI,‡ L.M. GUGLIOTTA,† and G.E. ELIÇABE ‡

† *Instituto de Desarrollo Tecnológico para la Industria Química (INTEC), CONICET and Univ. Nac. del Litoral, 3000 Santa Fe, Argentina*

jvega@ceride.gov.ar; lgug@intec.unl.edu.ar

‡ *Instituto de Investigación en Ciencia y Tecnología de Materiales (INTEMA), Univ. Nac. de Mar del Plata, 7600 Mar del Plata, Argentina.*

gfrontin@fi.mdp.edu.ar; elicabe@fi.mdp.edu.ar

Abstract— **This work deals with an ill-posed inverse problem in which a distribution function, $f(x)$, is estimated from two independent sets of non-negative relative measurements. Each measurement set is modeled through a Fredholm equation of the first kind, with unknown parameters in its kernel. While the first measurement model only includes a scalar unknown parameter, p_0, the second model contains a vector of unknown parameters, p. The proposed method consists of the following steps: (i) to obtain a first estimate of $f(x)$ and p_0 from the first measurement; (ii) to estimate the vector p from the second measurement and the previous estimate of $f(x)$; and (iii) to estimate an improved $f(x)$ by simultaneously using both measurements and the estimated parameters in a unique combined problem. The proposed algorithm is evaluated through a numerical example for simultaneously estimating the particle size distribution and the refractive index of a polymer latex, from combined measurements of elastic light scattering and turbidity.**

Keywords— **Inverse problem; parameter estimation; combined measurements; ELS; Turbidity.**

I. INTRODUCTION

Inverse problems are frequently present in most of the measurement systems that include non-ideal devices, or when only indirect measurements are available. Indirect measurements arise when a physical property of a sample, $f(x)$, must be estimated from the measurement of a different physical quantity, $g(y)$. The estimation problem consists in finding $f(x)$ from $g(y)$, on the basis of theoretical measurement models that relate these variables. Often, this kind of inverse problem is numerically 'ill-conditioned'; *i.e.*, small changes in the measured variable (for example, originated by different noise levels), may lead to large changes in the estimated variables (Kirsch, 1996). As a consequence, simulation of different $f(x)$ can generate almost the same $g(y)$, thus complicating the estimation problem.

Regularization methods replace the unstable original problem by a similar one, but stable or well-conditioned; and they usually include adjustable parameters, *a priori* knowledge of the solution, or some smoothness condition (Kirsch, 1996; Engl *et al*., 1996; Hansen, 1994). Often, when a regularization condition is included, then several solutions can be attained. Typically, a trade-off solution must be selected: (i) a strong regularization modifies the original problem, with the advantage of leading to a smooth solution; and (ii) a weak regularization keeps the original problem almost unchanged, but can originate oscillating solutions. In general, all regularization method includes (at least) an adjustable parameter, which is usually selected by the user on the basis of the obtained solutions. Alternatively, the regularization parameter can be automatically estimated through some numerical methods, as for example the Generalized Cross Validation (GCV) technique (Golub *et al*., 1979), or the L-curve technique (Hansen, 1994).

Combination of two or more independent sets of measurements (*e.g*., obtained from two different equipments), allows increasing the information content in the problem, and can contribute to improve the quality of the estimates. The numerical treatment of the combined problem is simple when the indirect measurements are absolute (*i.e*., when all proportionality constants are known), and the involved mathematical models are linear (Eliçabe and Frontini, 1996). However, if the model related to the measured physical quantities includes unknown proportionality constants, then the measurements are relative, and a 'normalization' parameter is required to adequately combine all sets of measurements in a unique problem (Frontini and Eliçabe, 2000). In such cases, the resulting combined problem may be nonlinear, and some alternative algorithms have been proposed to solve it. For example, Frontini and Eliçabe (2000) developed a method to estimate in a single step $f(x)$ and the normalization parameter. Alternatively, Vega *et al*. (2001) and Vega *et al*. (2003), proposed a "two-step" procedure that involves the resolution of two linear estimation problems. In the first step, the normalization parameter is estimated on the basis of conciliating the two independent measurements; and in the second step, the sought $f(x)$ is obtained after numerical inversion of the combined problem.

The estimation problem becomes more difficult when the mathematical model is not exactly known, or when some parameters are unknown. Typically, several linear inverse problems with unknown kernel parame-